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Heat-kernel coefficients of the Laplace operator on the 3-dimensional ball

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Abstract

We consider the heat-kernel expansion of the massive Laplace operator on the three dimensional ball with Dirichlet boundary conditions. Using this example, we illustrate a very effective scheme for the calculation of an (in principle) arbitrary number of heat-kernel coefficients for the case where the basis functions are known. New results for the coefficients $B_{\frac{5}{2}}, ..., B_5$ are presented.

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It is important to know the explicit form for the coefficients in the short-time expansion of the heat-kernel, $K(t)$, for some Laplacian-like operator on a d-dimensional manifold $\mathcal{M}$. In mathematics the interest stems for example from connections between the heat-equation and the Atiyah-Singer index theorem [1], whereas in physics the interest in this expansion lies for example in the domain of quantum field theory where it is commonly known as the (integrated) Schwinger-De Witt proper-time expansion [2].

If the manifold $\mathcal{M}$ has a boundary $\partial\mathcal{M}$, the coefficients B_n in the short time expansion have volume and boundary parts [3]. Thus

$$K(t) \sim (4\pi t)^{-\frac{d}{2}} \sum_{k=0,1/2,1,\ldots}^{\infty} B_k t^k \tag{1}$$

with

$$B_k = \int_{\mathcal{M}} dV\ b_n + \int_{\partial\mathcal{M}} dS\ c_n. \tag{2}$$

For the volume part effective systematic schemes have been developed (see for example [4]). The calculation of c_n is in general more difficult. Only relatively recently the coefficient c_2 for Dirichlet and Neumann boundary conditions have been found [5, 6]. When using the general formalism of ref. [5] for higher spin particles, Moss and Poletti [7] found a discrepancy with the direct calculations of D'Eath and Esposito [8] (see also [9]). The latter results have been confirmed in [10] where a new systematic scheme for the calculation of c_2 has been developed in the context of the Hartle-Hawking wave-function of the universe for the case when the full set of basis functions is known [10]. Finally, very recently the discrepancy have been resolved [11] and now the results found using the general algorithm [12] are in agreement with the direct calculations [8, 9, 10].

We will use a variant of the approach of ref. [10] in order to show that higher coefficients c_n may be calculated very effectively for the case that the basis functions are known. To illustrate the method in this letter we will concentrate on the calculation of the heat-kernel coefficients of the elliptic operator

$$(-\Delta + m^2)\phi_{n,l,m} = (\lambda_{n,l,m}^2 + m^2)\phi_{n,l,m} \tag{3}$$

in the three dimensional ball defined by $B_R = \{\vec{x} \in \mathrm{I\!R}^3, |\vec{x}| \leq R\}$. We will treat explicitly Dirichlet boundary conditions, $\phi_{n,l,m}(|\vec{x}| = R) = 0$. As will be clear afterwards, other boundary conditions and higher dimensional balls may be treated in exactly the same way.

Starting point is the equation

$$J_{l+\frac{1}{2}}(\lambda_{n,l,m} R) = 0 \tag{4}$$

for the eigenvalues $\lambda_{n,l,m}$, which are $(2l+1)$-times degenerated. We are especially interested in the calculation of the heat-kernel coefficients defined in equation (1). Instead of

calculating the heat-kernel coefficients itself, we will concentrate on the zeta function of the operator, eq. (3), and recover the heat-kernel coefficients using the relations [13]

$$Res\ \zeta(s) = \frac{B_{\frac{m}{2}-s}}{(4\pi)^{\frac{m}{2}}\Gamma(s)} \tag{5}$$

for $s = \frac{m}{2}, \frac{m-1}{2}, ..., \frac{1}{2}; -\frac{2l+1}{2}$, for $l \in \mathbb{N}_0$, and

$$\zeta(-p) = (-1)^p p! \frac{B_{\frac{m}{2}+p}}{(4\pi)^{\frac{m}{2}}} \tag{6}$$

for $p \in \mathbb{N}_0$.

Using relation (4) for the eigenvalues, one may write the zeta function in the form (for a similar procedure see [14])

$$\begin{aligned} \zeta(s) &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \lambda_{n,l,m}^{-2s} \\ &= \sum_{l=0}^{\infty}(2l+1)\int_{\gamma} \frac{dk}{2\pi i}(k^2+m^2)^{-s}\frac{\partial}{\partial k}\ln J_{l+\frac{1}{2}}(kR), \end{aligned} \tag{7}$$

where the contour γ is counterclockwise enclosing all eigenvalues which are known to be situated on the positive real axis. As it stands, the representation (7) is valid for $\Re s > \frac{3}{2}$.

Before considering in detail the l-summation, let us first construct an analytic continuation of the k-integral in equation (7) alone and let us define for that reason

$$\zeta_\nu(s) = \int_{\gamma} \frac{dk}{2\pi i}(k^2+m^2)^{-s}\frac{\partial}{\partial k}\ln J_\nu(kR), \tag{8}$$

with $\nu = l+1/2$. Deforming the contour to the imaginary axis, the analytic continuation,

$$\zeta_\nu(s) = \frac{\sin(\pi s)}{\pi}\int_{mR}^{\infty} dk \ \left(\left(\frac{k}{R}\right)^2 - m^2\right)^{-s}\frac{\partial}{\partial k}\ln I_{l+\frac{1}{2}}(k), \tag{9}$$

valid in the strip $\frac{1}{2} < \Re s < 1$, may be found. A similar representation valid for $m = 0$ has been given in [15].

In order to continue, the idea is to make use of the uniform expansion of the Bessel function $I_\nu(k)$ for $\nu \to \infty$ as $z = k/\nu$ fixed [16]. Actually we use the expansion

$$\frac{\partial}{\partial k}\ln I_\nu(k) \sim \frac{1}{\sqrt{1-t^2}}\sum_{n=0}^{\infty}\frac{d_n(t)}{\nu^n} \tag{10}$$

with $t = \frac{1}{\sqrt{1+z^2}}$. Here the functions $d_n(t)$ fulfill the recurrence relation

$$d_n(t) = \frac{1}{2}t(1-t^2)\left(t\frac{\partial}{\partial t} - 1\right)d_{n-1}(t) - \frac{1}{2}\sum_{k=1}^{n-1} d_k(t)d_{n-k}(t), \tag{11}$$

starting with $d_0(t)=1$. In order to calculate up to B_5 one needs the first eleven coefficients d_n, which can be easily calculated using the recurrence.

Adding and subtracting the leading terms of the asymptotic expansion for $\nu\to\infty$, eq. (9) may be split into two pieces,

$$\zeta_\nu(s)=N_\nu(s)+\sum_{i=1}^{N+1}A_\nu^i(s), \tag{12}$$

with

$$N_\nu(s)=\frac{\sin(\pi s)}{\pi}\int\limits_{mR/\nu}^{\infty}dk\ \left(\left(\frac{k\nu}{R}\right)^2-m^2\right)^{-s}\left\{\frac{\partial}{\partial k}\ln I_{l+\frac12}(k\nu)-\frac{1}{\sqrt{1-t^2}}\sum_{n=0}^{N}\frac{d_n(t)}{\nu^n}\right\} \tag{13}$$

and

$$A_\nu^i(s)=\frac{\sin(\pi s)}{\nu^{i-1}\pi}\int\limits_{mR/\nu}^{\infty}dk\ \left(\left(\frac{k\nu}{R}\right)^2-m^2\right)^{-s}\frac{\partial}{\partial k}d_{i-1}(t). \tag{14}$$

As it stands, the A_ν^i, equation (14), are well defined (at least) in the strip $1/2<\Re s<1$. However, the analytic continuation in the parameter s to the whole complex plane in terms of known function may be provided. To explain afterwards some details of the calculation let us give explicitly only the A_ν^i of the two leading terms in the asymptotics (10),

$$\begin{aligned}A_\nu^1(s) &= \frac{m^{-2s}\sin(\pi s)}{2\pi^{\frac32}}Rm\\ &\times \quad \Gamma\left(s-\frac12\right)\Gamma(1-s)\ {}_2F_1\left(-\frac12,s-\frac12;\frac12;-\left(\frac{\nu}{mR}\right)^2\right), \qquad (15)\\ A_\nu^2(s) &= -\frac{m^{-2s}\sin(\pi s)}{4\pi}\\ &\times \quad \Gamma(s)\Gamma(1-s)\ {}_2F_1\left(1,s;1;-\left(\frac{\nu}{mR}\right)^2\right). \qquad (16)\end{aligned}$$

Similar expressions for higher A_ν^i can be calculated, e.g. using standard integration packages. As mentioned, we did explicitly calculate the coefficients up to B_5 and thus needed A_ν^i for $i=1,...,11$. Here ${}_2F_1(a,b,c;z)$ denotes the hypergeometric function [17].

The representation (12) has the following very important properties. First of all, by considering the asymptotics of the integrand in equation (13) for $k\to mR/\nu$ and $k\to\infty$, it may be seen that

$$N(s)=\sum_{l=0}^{\infty}(2l+1)N_{l+\frac12}(s)$$

is analytic in the strip $1-N/2 < \Re s < 1$. For that reason it gives no contribution to the residue of $\zeta(s)$ in that strip. Furthermore, for $s=-k$, $k=0,1,2,3$, we have $N(s)=0$ and thus it does also not contribute to the values of the zeta function at those points. Together with eqs. (5), (6), this yields, that the heat-kernel coefficients are only determined by the terms $A_i(s)$ with

$$A_i(s) = \sum_{l=0}^{\infty} (2l+1) A^i_{l+\frac{1}{2}}(s). \tag{17}$$

However, the $A_i(s)$ may be given in terms of Hurwitz zeta functions and an explicit representation of $A_i(s)$ showing the meromorphic structure in the whole complex plane may be given. The sum in (17) may be easily done by means of the Mellin-Barnes type integral representation of the hypergeometric function,

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{\mathcal{C}} dt\ \frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)}{\Gamma(c+t)} (-z)^t, \tag{18}$$

where the contour $\mathcal{C}$ is such that the poles of $\Gamma(a+t)$ and $\Gamma(b+t)$ lie to the left of it and the poles of $\Gamma(-t)$ to the right [17]. Defining

$$h(a,b,c;n) = \sum_{l=0}^{\infty} \left(l+\frac{1}{2}\right)^n {}_2F_1\left(a,b;c;-\left(\frac{l+\frac{1}{2}}{\mu}\right)^2\right) \tag{19}$$

and closing the contour $\mathcal{C}$ to the left, we arrive at

$$\begin{aligned} h(a,b,c;n) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mu^{2k} \times \\ &\left(\mu^{2a} \frac{\Gamma(b-a-k)\Gamma(a+k)}{\Gamma(c-a-k)} \zeta_H\left(-n+2a+2k;\frac{1}{2}\right)\right. \\ &\left. +\mu^{2b} \frac{\Gamma(a-b-k)\Gamma(b+k)}{\Gamma(c-b-k)} \zeta_H\left(-n+2b+2k;\frac{1}{2}\right)\right), \end{aligned} \tag{20}$$

which may be used for all summations we need for the calculation of the A_i's. For example we obtain

$$\begin{aligned} A_1 &= \frac{m^{-2s}\sin(\pi s)}{\pi^{\frac{3}{2}}} Rm\Gamma\left(s-\frac{1}{2}\right)\Gamma(1-s) h\left(-\frac{1}{2}, s-\frac{1}{2}, \frac{1}{2}; 1\right) \\ &= R^{2s} \frac{\sin(\pi s)}{2\pi^{\frac{3}{2}}} \Gamma(1-s) \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} (mR)^{2l} \frac{\Gamma\left(l+s-\frac{1}{2}\right)}{s+l} \zeta_H\left(2l+2s-2;\frac{1}{2}\right), \\ A_2 &= -\frac{m^{-2s}\sin(\pi s)}{2\pi} \Gamma(s)\Gamma(1-s) h(1,s,1;1) \\ &= -R^{2s} \frac{\sin(\pi s)}{2\pi} \Gamma(1-s) \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} (mR)^{2l} \Gamma(l+s)\, \zeta_H\left(2l+2s-1;\frac{1}{2}\right), \end{aligned}$$

and similar expressions for the other A_i's, $i = 1, ..., 11$. The sums appearing in the $A_i(s)$ are convergent for $|mR| < 1/2$. Using this representation, equations (5), (6), together with

$$\begin{aligned}
\zeta_H\left(1+\epsilon;\frac{1}{2}\right) &= \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0), \\
\Gamma(\epsilon - n) &= \frac{1}{\epsilon}\frac{(-1)^n}{n!} + \mathcal{O}(\epsilon^0),
\end{aligned}$$

the heat-kernel coefficients may be easily determined. Summarizing, we find the following new results for the coefficients $B_{\frac{5}{2}}, ..., B_5$,

$$\begin{aligned}
B_{\frac{5}{2}} &= \pi^{\frac{3}{2}}\left(\frac{m^2}{6} - \frac{1}{120R^2} - m^4R^2\right) \\
B_3 &= \pi\left(-\frac{64}{9009R^3} + \frac{16m^2}{315R} + \frac{4m^4R}{3} - \frac{2m^6R^3}{9}\right) \\
B_{\frac{7}{2}} &= \pi^{\frac{3}{2}}\left(-\frac{m^4}{12} - \frac{47}{20160R^4} + \frac{m^2}{120R^2} + \frac{m^6R^2}{3}\right) \\
B_4 &= \pi\left(-\frac{202816}{72747675R^5} + \frac{64m^2}{9009R^3} - \frac{8m^4}{315R}\right. \\
&\quad \left. -\frac{4m^6R}{9} + \frac{m^8R^3}{18}\right) \\
B_{\frac{9}{2}} &= \pi^{\frac{3}{2}}\left(\frac{m^6}{36} - \frac{521}{443520R^6} + \frac{47m^2}{20160R^4}\right. \\
&\quad \left. -\frac{m^4}{240R^2} - \frac{m^8R^2}{12}\right) \\
B_5 &= \pi\left(-\frac{25426048}{15058768725R^7} + \frac{202816m^2}{72747675R^5} - \frac{32m^4}{9009R^3}\right. \\
&\quad \left. +\frac{8m^6}{945R} + \frac{m^8R}{9} - \frac{m^{10}R^3}{90}\right)
\end{aligned}$$

For $m = 0$ the coefficient $B_{5/2}$ agrees with the one found by Kennedy [18].

As mentioned, other boundary conditions, higher dimensional balls, and even higher coefficients may be found in exactly the same way without any additional complication. This and more details of the calculation will be given in a separate publication.

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